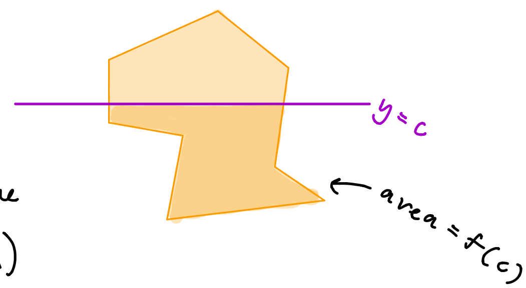
Antipodes

Motivation: If $A \subseteq \mathbb{R}^2$ is a bounded region w/ boundary made up of finitely many line segments, we can always find a line that perfectly bisects it:

Let $f(c)$ = area of A below $y = c$.



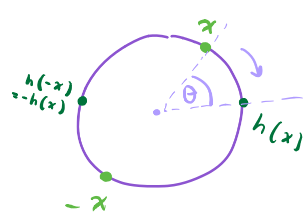
f is continuous, so the intermediate value theorem says $\exists\, c$ s.t. $f(c) = \frac{1}{2}(\text{area of } A)$

What if instead we have 2 regions? Can we find a line that bisects both simultaneously? It turns out we can, by a corollary to the Borsuk-Ulam theorem.

First we need some definitions and theorems...

Def: If $x \in S^n$, its <u>antipode</u> is the point $-x$. A map $h: S^n \to S^m$ is <u>antipode-preserving</u> if $h(-x) = -h(x)$ for all $x \in S^n$.

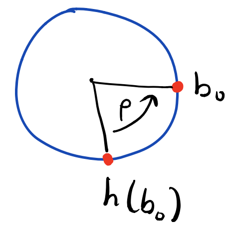
Ex: Rotation of S^1 by θ is antipode-preserving.



Thm: If $h: S^1 \to S^1$ is continuous and antipode-preserving, then it's <u>not</u> nulhomotopic.

Pf: Let $b_0 = (1,0) \in S^1$. Let $\rho: S^1 \to S^1$ be the rotation mapping $h(b_0)$ to b_0.

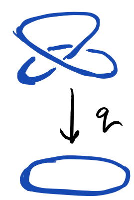
Since h is antipode-preserving and ρ is antipode-preserving, $\rho \circ h$ is antipode-preserving.



If H were a homotopy between h and a constant map, then $\rho \circ H$ would be a homotopy between $\rho \circ h$ and a constant map. So we can reduce to the case $h(b_0) = b_0$.

Let $q: S^1 \to S^1$ be the deg 2 cover $q(\cos\theta, \sin\theta) = (\cos 2\theta, \sin 2\theta)$

This is a covering map and thus a quotient map.



Note that two antipodal points (π apart) get sent to the same pt.

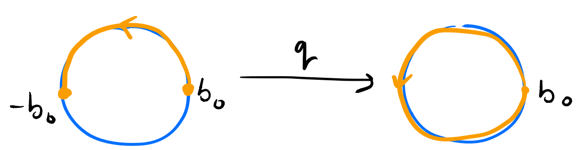
so $q: S^1 \to S^1/\sim$, where $z \sim -z$.

So $q(h(-z)) = q(-h(z)) = q(h(z))$. i.e. $q \circ h$ sends the points that get identified via q to the same point.

Thus, $\exists$ $k: S^1 \to S^1$ s.t. $q \circ h = k \circ q$.

$$\begin{tikzcd} S^1 \arrow[r, "h"] \arrow[d, "q"'] & S^1 \arrow[d, "q"] \\ S^1 \arrow[r, "k"] & S^1 \end{tikzcd}$$

Note that k_* is nontrivial: let $\tilde{f}$ be a path from b_0 to $-b_0$



Let $f = q \circ \tilde{f}$.

$k_*[f] = [k \circ (q \circ \tilde{f})] = [q \circ (h \circ \tilde{f})]$.

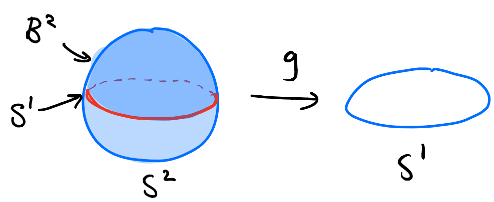
Since h is antipode-preserving, $h \circ \tilde{f}$ is a path from b_0 to $-b_0$. This is a lift of the loop $q \circ (h \circ \tilde{f})$ which doesn't end at b_0, so by the lifting correspondence, $k_*[f]$ is not the identity, so k_* is nontrivial, and thus it's injective (from $\mathbb{Z}$ to $\mathbb{Z}$).

q_* is also injective (mult by 2). So $k_* \circ q_* = q_* \circ h_*$ is injective, so h_* is $\Rightarrow$ h is not nullhomotopic. $\square$

Cor: There is no continuous antipode-preserving map $g: S^2 \to S^1$.

Pf: Suppose g is continuous and antipode-preserving.

Consider $S^1 \subseteq S^2$ as the equator.



Then $g|_{S^1}$ is not nullhomotopic, but it extends to a map on the ball B^2, which is a contradiction. $\square$

Borsuk-Ulam Theorem for S^2: If $f: S^2 \to \mathbb{R}^2$ is continuous, there's a point $x \in S^2$ s.t. $f(x) = f(-x)$.

Pf: Suppose $f(x) \neq f(-x)$ for any x. Then

$$g(x) = \frac{f(x) - f(-x)}{\| f(x) - f(-x) \|}$$ is a continuous map from S^2 to S^1, and

$g(x) = -g(-x)$ $\forall x$, a contradiction. $\square$

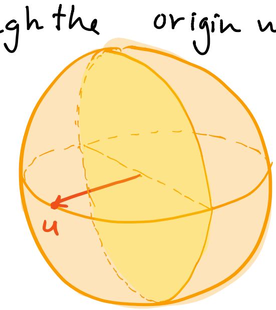
Cor: An open set in $\mathbb{R}^2$ cannot be homeomorphic to an open set in $\mathbb{R}^n$ for $n \geq 3$.

Note: this is obvious for $\mathbb{R}^1$ and any other dim since removing a point from (a,b) disconnects it. This is less obvious in higher dims.

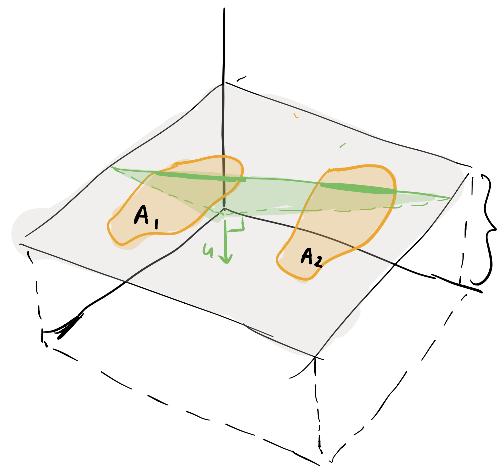
Pf: If $U \subseteq \mathbb{R}^n$ open, and $f: U \to \mathbb{R}^2$ an embedding, then there is some $\overline{B_r(x)} \subseteq U$ for r small, but $\overline{B_r(x)}$ is homeomorphic to $B^n \supseteq B^3 \supseteq S^2$, so we get a map $f|_{S^2}: S^2 \to \mathbb{R}^2$ which is a homeo. onto its image, a contradiction.

Thm: Given two bounded (polygonal) regions in $\mathbb{R}^2$, there's a line in $\mathbb{R}^2$ that bisects each of them.

Pf: Let A_1 and A_2 lie in the plane $\mathbb{R}^2 \times \{1\}$ in $\mathbb{R}^3$. Given $u \in S^2$, let $P \subseteq \mathbb{R}^3$ be the plane through the origin w/ u as a normal vector.



P divides $\mathbb{R}^3$ into to half-planes. Let $f_i(u)$ = area of A_i that lies on the side of the line in the direction of u.



Thus $f_i(u) + f_i(-u) = \text{area of } A_i$.

Then $F(u) = (f_1(u), f_2(u))$ is a continuous map from S^2 to $\mathbb{R}^2$.

$\Rightarrow \exists$ some u s.t. $F(u) = F(-u)$

$\Rightarrow f_i(u) = \frac{1}{2}(\text{area } A_i)$. $\square$

In fact, this generalizes to n bounded measurable sets in $\mathbb{R}^n$. In $\mathbb{R}^3$, it's called the "ham sandwich theorem".